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Robust Inflation-Forecast-Based Rules to Shield against Indeterminacy*

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Abstract

This paper provides a first attempt to quantify and at the same time utilize estimated measures of uncertainty for the design of robust interest rate rules. We estimate several variants of a linearized form of a New Keynesian model using quarterly US data. Both our theoretical and numerical results indicate that Inflation-Forecast-Based (IFB) rules are increasingly prone to the problem of indeterminacy as the forward horizon increases. As a consequence the stabilization performance of optimized rules of this type worsens too. Robust IFB rules can be designed to avoid indeterminacy in an uncertain environment, but at an increasing utility loss as rules become more forward-looking.

JEL Classification: E52, E37, E58

Keywords: robustness, Taylor rules, inflation-forecast-based rules, indeterminacy

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Contents

1 Introduction

"Uncertainty is not just an important feature of the monetary policy landscape; it is the defining characteristic of that landscape." Alan Greenspan[1]

This paper adopts a consistently Bayesian approach to the measurement of uncertainty and the design of robust rules for the conduct of monetary policy. Employing a closed economy New Keynesian model, the sources of uncertainty in our paper are the structural parameters and the volatility of the white noise disturbances. We estimate several variants of a linearized form of the model using quarterly US data. From these competing specifications we obtain estimates for posterior model probabilities and, for each model variant, estimates of the posterior densities of the parameters.

Using these rival models with estimates set at their median values and the estimated probabilities we then design rules that are robust in two senses: '*weakly robust*' rules are guaranteed to be stable and determinate in all the possible central variants of the model whereas '*strongly robust*' rules, also guarantee stable and unique equilibria and, in addition, use the probabilities to minimize an expected loss function of the central bank subject to this model uncertainty. Both these forms of robustness across models with estimates at their median values we refer to as 'M-robustness', weak or strong. A more demanding robustness requirement is minimize the expected loss across *all* possible parameter values drawn from a large sample constructed using the estimated posterior parameter distributions as well as the model probabilities. This we refer to as 'P-robustness', weak or strong. Table 1 summarizes this taxonomy.

	Weak Robustness	Strong Robustness
M (Model)-Robustness	(M,W)	(M,S)
P (Parameter)-Robustness	(P,W)	(P,S)

Table 1. Four Robustness Criteria

The monetary rules studied in the paper are defined in terms of feedback parameters. Weakly robust rules then define a space of these parameters for which stability and determinacy is guaranteed across models with model parameters at median values, or across

[1]Federal Reserve Bank of Kansas (2003), Opening Remarks.

all possible parameter values. In each case, a strongly robust rule chooses from the set of weakly robust rules the rule that maximizes the policymaker's expected utility.

Our approach thus differs from existing work on the design of robust policy rules in a number of important respects. First, existing work that assumes unstructured model uncertainty typically posits the latter by arbitrarily calibrating the relative probability of alternative models being true representations of the economy (see for example Angeloni *et al.* (2003); Coenen (2003); Levin *et al.* (2003)).[2] This paper provides a first attempt to quantify and at the same time utilize *estimated* measures of uncertainty for the design of robust rules. Second, the literature taking the rival model approach typically confines itself to what we call strong M-robustness. Third, we examine robust policy in a unified framework that compares different simple rules with each other, and with their optimal counterparts.

Throughout we focus on Taylor-type rules, and in particular on inflation-forecast-based (IFB) rules. These are 'simple' rules as in Taylor (1993), but where the policy instrument responds to deviations of expected, rather than current inflation from target. In most applications, the inflation forecasts underlying IFB rules are taken to be the endogenous rational-expectations forecasts conditional on an intertemporal equilibrium of the model. These rules are of interest because, as shown in Clarida *et al.* (2000) and Castelnuovo (2003), estimates of IFB-type rules appear to be a good fit to the actual monetary policy in the US and Europe of recent years. However, with IFB rules indeterminacy can be particularly severe and can take two forms: if the response of interest rates to a rise in expected inflation is insufficient, then real interest rates fall, thus raising demand and confirming any exogenous expected inflation. But indeterminacy is also possible if the rule is overly aggressive (Bernanke and Woodford (1997); Batini and Pearlman (2002); Giannoni and Woodford (2002); Batini *et al.* (2004), BLP hereafter).

We find four main results. First, in each of our three model variants with the highest posterior model probabilities chosen for the policy exercise, there are significant gains from stabilization using an optimized inflation targeting rule with the interest rate feeding back on current inflation. Second, a strongly M-robust and P-robust current inflation rule

[2]This literature contrasts with the minmax framework of Hansen and Sargent (2002) that assumes unstructured model uncertainty. Walsh (2003) provides a useful overview of this approach and Tetlow and von zur Muehlen (2002) provides a comparison.

can be designed that achieves almost all of the stabilization gain that would be achieved if there was no model uncertainty. Third, integral interest rate rules where the change in interest rates feeds back on current of expected future inflation perform better than non-integral rules.[3] Fourth, the optimized inflation targeting rules perform increasingly less well as the forward horizon increases from $j = 0$ (the current inflation rule) to $j = 4$ quarters. Denoting such a rule by IFBj, we find a qualitative difference between IFB0, IFB1 rules on the one hand and IFBj, $j \geq 2$ rules. For IFB0 and IFB1 optimized rules, little by way of utility outcome is lost by insisting on M-robustness or P-robustness. For the IFB3 and IFB4 rule, robustness in both senses is only achieved by sacrificing the utility outcome when each of the models in turn describes the true economic environment. This deterioration is especially marked if we insist on P-robustness as our design criterion.

The rest of the paper is organized as follows. Section 2 sets out our model. Section 3 provides a theoretical examination of the indeterminacy problem of IFB rules using the root locus method employed by Batini and Pearlman (2002) and BLP. This analysis indicates which features of the model and the rule make them indeterminacy-prone. Section 4 first focuses on optimized IFB rules and optimal rules without uncertainty before we turn to the robust policy problem in section 5. Section 6 concludes the paper.

2 The Model

Our model is the closed economy version of BLP. There is one traded risk-free nominal bond. A final homogeneous good is produced competitively using a CES technology consisting of a continuum of differentiated non-traded goods. Intermediate goods producers and household suppliers of labor have monopolistic power. Nominal prices of intermediate goods are sticky. We incorporate a bias for consumption of home-produced goods, habit formation in consumption, and Calvo price setting with indexing of prices for those firms who, in a particular period, do not re-optimize their prices. The latter two aspects of the model follow Christiano *et al.* (2001) and, as with these authors, our motivation is an empirical one: to generate sufficient inertia in the model so as to enable it, in calibrated form, to reproduce commonly observed output, inflation and nominal interest rate responses to exogenous shocks. Our model is stochastic with two exogenous AR(1) stochastic processes

[3] This accords the results of Levin *et al.* (2003).

for total factor productivity in the intermediate goods sector and government spending.

2.1 Households

A representative household r maximizes

$$\mathcal{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{(C_t(r) - H_t)^{1-\sigma}}{1-\sigma} + \chi \frac{\left(\frac{M_t(r)}{P_t}\right)^{1-\varphi}}{1-\varphi} - \kappa \frac{N_t(r)^{1+\phi}}{1+\phi} + u(G_t) \right] \tag{1}$$

where $\mathcal{E}_t$ is the expectations operator indicating expectations formed at time t, $C_t(r)$ is an index of consumption, $N_t(r)$ are hours worked, H_t represents the habit, or desire not to differ too much from other consumers, and we choose it as $H_t = hC_{t-1}$, where C_t is the average consumption index and $h \in [0, 1)$ and $\sigma > 1$ is a risk aversion parameter. $M_t(r)$ are end-of-period nominal money balances and $u(G_t)$ is the utility from exogenous real government spending G_t.

The representative household r must obey a budget constraint:

$$P_tC_t(r) + D_t(r) + M_t(r) = W_t(r)N_t(r) + (1 + i_{t-1})D_{t-1}(r) + M_{t-1}(r) + \Gamma_t(r) - P_t\tau_t \tag{2}$$

where P_t is a price index, $D_t(r)$ are end-of-period holdings of riskless nominal bonds with nominal interest rate i_t over the interval $[t, t+1]$. $W_t(r)$ is the wage, $\Gamma_t(r)$ are dividends from ownership of firms and τ_t are lump-sum real taxes. In addition, if we assume that households' labour supply is differentiated with elasticity of supply η, then (as we shall see below) the demand for each consumer's labor is given by

$$N_t(r) = \left(\frac{W_t(r)}{W_t} \right)^{-\eta} N_t \tag{3}$$

where $W_t = \left[\int_0^1 W_t(r)^{1-\eta} dr \right]^{\frac{1}{1-\eta}}$ is an average wage index and N_t is average employment.

Maximizing (1) subject to (2) and (3) and imposing symmetry on households (so that $C_t(r) = C_t$, etc) yields standard results:

$$1 = \beta(1 + i_t)\mathcal{E}_t \left[\left(\frac{C_{t+1} - H_{t+1}}{C_t - H_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}} \right] \tag{4}$$

$$\left(\frac{M_t}{P_t} \right)^{-\varphi} = \frac{(C_t - H_t)^{-\sigma}}{\chi P_t} \left[\frac{i_t}{1 + i_t} \right] \tag{5}$$

$$\frac{W_t}{P_t} = \frac{\kappa}{(1 - \frac{1}{\eta})} N_t^{\phi} (C_t - H_t)^{\sigma} \tag{6}$$

(4) is the familiar Keynes-Ramsey rule adapted to take into account of the consumption habit. In (5), the demand for money balances depends positively on consumption relative to habit and negatively on the nominal interest rate. Given the central bank's setting of the latter, (5) is completely recursive to the rest of the system describing our macro-model and will be ignored in the rest of the paper. (6) reflects the market power of households arising from their monopolistic supply of a differentiated factor input with elasticity η.

2.2 Firms

Competitive final goods firms use a continuum of non-traded intermediate goods according to a constant returns CES technology to produce aggregate output

$$Y_t = \left(\int_0^1 Y_t(m)^{(\zeta-1)/\zeta} dm \right)^{\zeta/(\zeta-1)} \tag{7}$$

where ζ is the elasticity of substitution. This implies a set of demand equations for each intermediate good m with price $P_t(m)$ of the form

$$Y_t(m) = \left(\frac{P_t(m)}{P_t} \right)^{-\zeta} Y_t \tag{8}$$

where $P_t = \left[\int_0^1 P_t(m)^{1-\zeta} dm \right]^{\frac{1}{1-\zeta}}$. P_t is an aggregate intermediate price index, but since final goods firms are competitive and the only inputs are intermediate goods, it is also the domestic price level.

In the intermediate goods sector each good m is produced by a single firm m using only differentiated labour with another constant returns CES technology:

$$Y_t(m) = A_t \left(\int_0^1 N_t(r,m)^{(\eta-1)/\eta} dr \right)^{\eta/(\eta-1)} \tag{9}$$

where $N_t(r,m)$ is the labour input of type r by firm m and A_t is an exogenous shock capturing shifts to trend total factor productivity (TFP) in this sector. Minimizing costs $\int_0^1 W_t(r) N_t(r,m) dr$ and aggregating over firms and denoting $\int_0^1 N_t(r,m) dm = N_t(r)$ leads to the demand for labor as shown in (3). In an equilibrium of equal households and firms, all wages adjust to the same level W_t and it follows that $Y_t = A_t N_t$.

For later analysis it is useful to define the real marginal cost as the wage relative to domestic producer price. Using (6) and $Y_t = A_t N_t$ this can be written as

$$MC_t \equiv \frac{W_t}{A_t P_t} = \frac{\kappa}{(1-\frac{1}{\eta})A_t} \left(\frac{Y_t}{A_t} \right)^{\phi} (C_t - H_t)^{\sigma} \tag{10}$$

Now we assume that there is a probability of $1-\xi$ at each period that the price of each intermediate good m is set optimally to $P_t^0(m)$. If the price is not re-optimized, then it is indexed to last period's aggregate producer price inflation.[4] With indexation parameter $\gamma \geq 0$, this implies that successive prices with no re-optimization are given by $P_t^0(m)$, $P_t^0(m)\left(\frac{P_t}{P_{t-1}}\right)^\gamma$, $P_t^0(m)\left(\frac{P_{t+1}}{P_{t-1}}\right)^\gamma$, ... For each intermediate producer m the objective is at time t to choose $\{P_t^0(m)\}$ to maximize discounted profits

$$\mathcal{E}_t \sum_{k=0}^{\infty} \left(\frac{\xi}{1+i_t}\right)^k Y_{t+k}(m) \left[P_t^0(m)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^\gamma - \frac{W_{t+k}}{A_{t+k}}\right] \tag{11}$$

given i_t (since firms are atomistic), subject to (8). The solution to this is

$$\mathcal{E}_t \sum_{k=0}^{\infty} \left(\frac{\xi}{1+i_t}\right)^k Y_{t+k}(m) \left[P_t^0(m)\left(\frac{P_{t+k-1}}{P_{t-1}}\right)^\gamma - \frac{1}{(1-1/\zeta)}\frac{W_{t+k}}{A_{t+k}}\right] = 0 \tag{12}$$

and by the law of large numbers the evolution of the price index is given by

$$P_{t+1}^{1-\zeta} = \xi\left(P_t\left(\frac{P_t}{P_{t-1}}\right)^\gamma\right)^{1-\zeta} + (1-\xi)(P_{t+1}^0)^{1-\zeta} \tag{13}$$

2.3 Equilibrium

In equilibrium, goods markets, money markets and the bond market all clear. Equating the supply and demand of the consumer good we obtain

$$Y_t = A_t N_t = C_t + G_t \tag{14}$$

A balanced budget government budget constraint

$$G_t = \tau_t + \frac{M_t - M_{t-1}}{P_t} \tag{15}$$

completes the model. Given interest rates i_t (expressed later in terms of an optimal or IFB rule) the money supply is fixed by the central banks to accommodate money demand. By Walras' Law we can dispense with the bond market equilibrium condition and therefore the government budget constraint that determines taxes τ_t. Then the equilibrium is defined at $t=0$ by stochastic processes C_t, D_t, P_t, M_t, W_t, Y_t, N_t, given past price indices and exogenous TFP and government spending processes.

[4]Thus we can interpret $\frac{1}{1-\xi}$ as the average duration for which prices are left unchanged.

2.4 Linearization and State Space Representation

We now linearize about the deterministic zero-inflation steady state. Output is then at its sticky-price, imperfectly competitive natural rate and from the Keynes-Ramsey condition (4) the nominal rate of interest is given by $\bar{\imath} = \frac{1}{\beta} - 1$. Define all lower case variables as proportional deviations from this baseline steady state.[5] Then the linearization takes the form:

$$\begin{aligned}
\pi_t &= \frac{\beta}{1+\beta\gamma}\mathcal{E}_t\pi_{t+1} + \frac{\gamma}{1+\beta\gamma}\pi_{t-1} + \frac{(1-\beta\xi)(1-\xi)}{(1+\beta\gamma)\xi}mc_t && (16)\\
mc_t &= -(1+\phi)a_t + \frac{\sigma}{1-h}(c_t - hc_{t-1}) + \phi y_t && (17)\\
c_t &= \frac{h}{1+h}c_{t-1} + \frac{1}{1+h}\mathcal{E}_t c_{t+1} - \frac{1-h}{(1+h)\sigma}(i_t - \mathcal{E}_t\pi_{t+1}) && (18)\\
y_t &= \frac{\bar{C}}{\bar{Y}}c_t + \frac{\bar{G}}{\bar{Y}}g_t && (19)\\
g_t &= \rho_g g_{t-1} + \epsilon_{gt} && (20)\\
a_t &= \rho_a a_{t-1} + \epsilon_{at} && (21)
\end{aligned}$$

Variables y_t, c_t, mc_t, a_t, g_t are proportional deviations about the steady state. $[\epsilon_{gt}, \epsilon_{at}]$ are i.i.d. disturbances. π_t and i_t are absolute deviations about the steady state. For later use we require the *output gap* the difference between output for the sticky price model obtained above and output when prices are flexible, y_{nt} say. The latter, obtained by setting $\xi = 0$ in (16) to (19), is in deviation form given by[6]

$$\begin{aligned}
\frac{\sigma}{1-h}(c_{nt} - hc_{n,t-1}) + \phi y_{nt} &= (1+\phi)a_t && (22)\\
y_{nt} &= \frac{\bar{C}}{\bar{Y}}c_{nt} + \frac{\bar{G}}{\bar{Y}}g_t && (23)
\end{aligned}$$

We can write this system in state space form as

$$\begin{bmatrix} \mathsf{z}_{t+1} \\ \mathcal{E}_t\mathsf{x}_{t+1} \end{bmatrix} = A\begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} + Bi_t + C\begin{bmatrix} \epsilon_{gt+1} \\ \epsilon_{at+1} \end{bmatrix} \qquad (24)$$

$$\begin{bmatrix} y_t \\ y_{nt} \end{bmatrix} = E\begin{bmatrix} \mathsf{z}_t \\ \mathsf{x}_t \end{bmatrix} \qquad (25)$$

[5]That is, for a typical variable X_t, $x_t = \frac{X_t - \bar{X}}{\bar{X}} \simeq \log\left(\frac{X_t}{\bar{X}}\right)$ where $\bar{X}$ is the baseline steady state. The interest rate however is now expressed as an absolute deviation about $\bar{\imath}$.

[6]Note that the zero-inflation steady states of the sticky and flexi-price steady states are the same.

where $\mathsf{z}_t = [a_t, g_t, c_{t-1}, c_{n,t-1}, \pi_{t-1}]$ is a vector of predetermined variables at time t and $\mathsf{x}_t = [c_t, \pi_t]$ are non-predetermined variables. Rational expectations are formed assuming an information set $\{z_s, x_s\}$, $s \leq t$, the model and the monetary rule.

2.5 Estimation

2.5.1 Overview

In this section we estimate four main variants of model (16)-(21) using Bayesian methods. In particular, we estimate: the most general specification of the model with both inflation and habit persistence (we label this variant '*Z*'); a version of the model without inflation persistence but with persistence in habits ($\gamma = 0$, variant '*G*'); a version without habit persistence but with persistence in inflation ($h = 0$, variant '*H*'); and finally a version with neither inflation nor habit persistence ($\gamma = h = 0$, variant '*GH*'). We close the model with a 1-quarter ahead IFB rule of the form (28) that is the subject of the next section.

Bayesian estimation of the model has the specific advantage that it provides a posterior distribution of the parameter values that allows us to make probabilistic statements about the functionals of the model(s)' parameters. Furthermore, it provides us with the odds on models that allow us to quantify how likely it is that the data would have come from a model with both habit and inflation persistence as opposed to a framework with just one of these mechanisms or neither. In this sense the estimation method *per se* supplies us with a consistent measure of both parameter (posterior distribution of the parameters) and model (posterior odds) uncertainty.[7]

The sub-sections below offer: a brief sketch of the methods used in estimation (subsection 2.5.2); a discussion of the specification of the prior distributions (sub-section 2.5.3); the results from the estimation of our four model specifications (sub-section 2.5.4); and a formal comparison of models (sub-section 2.5.5). This sub-section shows how we obtain the posterior model probabilities that we use as weights for the competing model specifications in the analysis of robust IFB rules under uncertainty.

[7] Justiniano and Preston (2004) discuss the many additional advantages of using Bayesian methods to estimate dynamic stochastic general equilibrium models. These include overcoming convergence problems with numerical routines to maximize the likelihood as well as providing measures of uncertainty that need not assume a symmetric distribution.

2.5.2 Methodology

Each model indexed by k and denoted $m_{k,}$ has an associated set of unknown parameters $\omega_k \in \Omega_k$. Following a Bayesian approach, our aim is to characterize the posterior distribution of the models' parameters, $p\left(\omega_k|Y^T, m_k\right)$,where Y^T stands for the full sample of observed data (T denotes the number of observations). Having specified a (perhaps model specific) prior density, $p(\omega_k|m_k)$, the posterior of the parameters is given by

$$p\left(\omega_k|Y^T, m_k\right) = \frac{\mathcal{L}\left(\omega_k|Y^T, m_k\right) p(\omega_k|m_k)}{\int \mathcal{L}\left(\omega_k|Y^T, m_k\right) p(\omega_k|m_k) d\omega_k} \tag{26}$$

where $\mathcal{L}\left(\omega_k|Y^T, m_k\right)$ is the likelihood obtained under the assumption of normally distributed disturbances from the state-space representation implied by the solution of the linear rational expectations model. The denominator in equation (26) corresponds to the marginal likelihood (also known as the 'marginal data density') and, as explained later, plays a key role in model comparisons.

The solution of the model is a non-linear function of the parameters which does not allow for any closed-form expression for the posterior density. Furthermore, the high dimension of the parameters space renders numerical integration inefficient. Markov Chain Monte Carlo (MCMC) methods, however, provide a feasible and accurate approximation to this density.

Following Schorfheide (2000) the estimation follows a two step approach. In the first step, a numerical algorithm is used to approximate the posterior mode by combining the likelihood $\mathcal{L}(Y^T|\omega_k, m_k)$ with the prior. In the second step, the obtained posterior mode is then used as starting value (ω_k^0) for a Random Walk Metropolis algorithm that generates draws from the posterior $p(\omega_k|Y^T, m_k)$. At each step i of the Markov Chain, the proposal density used to draw a new candidate parameter ω_k^* is a normal centered at the current state of the chain, $N(\omega_k^i, c\Sigma_k)$. A new draw is then accepted with probability

$$\alpha = \min(1, \frac{\mathcal{L}(Y^T|\omega_k^*, m_k)p(\omega_k^*|m_k)}{\mathcal{L}(Y^T|\omega_k^i, m_k)p(\omega_k^i|m_k)})$$

If accepted, $\omega_k^{i+1} = \omega_k^*$; otherwise, $\omega_k^{i+1} = \omega_k^\imath$. This may be viewed as a stochastic climbing algorithm. Whenever a new draw results in higher posterior probability than the current state of the chain, the draw is retained. Otherwise, there is probability (α) that you will jump to a point of lower posterior density. We generate chains of 130,000 draws in this manner discarding the first 30,000 iterations.[8]

Point estimates of the parameters ω_k can be obtained from the generated values by using various location measures, such as mean or, as in this paper, medians. Similarly, measures of uncertainty follow from computing the percentiles of the draws.

2.5.3 Data and Priors

We estimate the model(s) using quarterly US data on real GDP (detrended–as standard in the literature we detrend this using a Hodrick-Prescott filter, see Lubik and Schorfheide (2003), Juillard *et al.* (2004)), the Federal Funds rate (annualized, in percentage points), and the annualized log difference of the consumer price index (CPI) for the sample 1984:I-2003:IV.[9] All series were obtained from DataStream International.

Following Lubik and Schorfheide (2004) rather than de-meaning the series, we estimate the mean of inflation and the (unobservable) real interest rate, π^* and r^* respectively, together with the model(s) parameters. In turn, this gives the following mapping between observables (superscript *obs*) and the variables following the solution of the model.

$$\begin{pmatrix} \pi_t^{obs} \\ y_t^{obs} \\ i_t^{obs} \end{pmatrix} = \begin{pmatrix} \pi^* \\ 0 \\ \pi^* + r^* \end{pmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{pmatrix} \pi_t \\ y_t \\ i_t \end{pmatrix}$$

In addition, the mean of the real rate gives us an estimate of the discount factor $\beta = 1/\sqrt[4]{1 + \frac{r^*}{100}}$.

To proceed with the Bayesian estimation we need a prior distribution for the parameters. Details on our priors are presented in **Table B1** in Appendix B reporting the type of density, mean and standard deviation for each coefficient.[10] The last two columns also provide the 1% and 99% percentiles of the prior ordinates. In choosing these densities we considered the entire spectrum of prior existing empirical estimates or calibrations. As a

[8] This initial burn-in phase is intended to remove any dependence of the chain from its starting values.

[9] 8 observations, corresponding to the period 1982:I - 1983:IV are used to initialize the Kalman filter.

[10] In principle, is would be possible to specify flat or non-informative priors for estimating θ_k. However, in addition to being able to choose priors based on coefficients values available in the literature, flat priors are not well suited for model comparisons.

result, some of our priors are more widely dispersed, and therefore less tight than those chosen by other authors. [11]

The degree of habit formation (h), price indexation (γ) and interest smoothing in the IFB-type rule (ρ), as well as the autoregressive coefficients of the shocks (ρ_g and ρ_a) are all constrained to the unit interval, motivating our choice of Beta densities for these priors. The priors for h and γ are centered at 0.7, on the assumption that output and inflation are considerably inertial, in line with findings by Fuhrer and Moore (1995), Fuhrer (2000), Banerjee and Batini (2003, 2004) and Smets and Wouters (2004)(SW, 2004), among others. Likewise, our prior for the mean of ρ is rather high and close to the estimates from Clarida *et al.* (2000) (CGG,2000).

Priors for σ and ϕ are shaped in the form of a Gamma density and are chosen to be fairly flat, reflecting the wide dispersion of existing empirical estimates and calibrations of these parameters in the literature. (see Nelson and Nikolov (2002)).

The slope of the Phillips' curve, $\chi = \frac{(1-\beta\xi)(1-\xi)}{(1+\beta\gamma)\xi}$ is a function of the degree of price stickiness in the economy, ξ, and the discount factor. So we selected the prior for χ in line with the assumption that the quarterly discount factor is equal to 0.99 and prices are sticky for three quarters, as suggested by survey evidence on the average duration of US price contracts (see, for example, Blinder *et al.* (1998).[12]

Finally, the prior for θ accounts for the breadth of the spectrum of estimated responses to expected inflation by the US Federal Reserve. More specifically, our specification contains the 90% posterior intervals of Lubik and Schorfheide (2004)[13] and is looser than the prior specified by SW for the same parameter.[14]

[11] Throughout the estimation of different models, the share of government expenditures in output is calibrated at 0.22,which represents the sample average of this coefficient for our sample.

[12] It is worth noting that the results of the estimation from assuming a prior directly on the Calvo coefficient ξ are somewhat different. This may be because with a prior on lambda, as we have used now, the link between ξ and the discount factor in determining the slope of the PC is not imposed. We plan to re-run the estimation with this alternative prior as a robustness check.

[13] Note that in contrast to these authors however we constrain the estimation to the region of determinacy and therefore truncate the prior for θ. The results of their paper suggest, however, that at least for a Taylor rule on current inflation, indeterminacy has not been an issue for our sample. In light of the results in BLP, exploring whether their results extend to the estimation of IFB is left for a future project.

[14] In their paper, however, SW include the output gap in the Taylor rule.

2.5.4 Estimation Results

Table B2 in Appendix B, summarizes the results of estimating the four model variants (G,H,GH and Z). The three columns for each specification report the median, 1st and 9th decile of the 100,000 draws generated using the Random Walk-Metropolis algorithm used to approximate the posterior densities.

A few important things emerge from the table. First, estimates of the policy coefficients are fairly robust across specifications. Posterior estimates of ρ are tightly concentrated on values that suggest a substantial degree of interest smoothing, in accordance with results reported by CGG amongst other authors. Meanwhile, the posterior density for θ is remarkably similar (that is both in medians and percentiles) across the first three specifications, implying a very aggressive response by the US Federal Reserve to expected inflation, in line with findings by CGG for a similar rule and sample.

The median estimates for r^* translate into a median value of 0.995 for the stochastic discount factor which, in turn, implies plausible estimates for the degree of price stickiness based on the inferred values for χ. The implied point estimates of ξ range from 0.36 up to 0.67, increasing, as expected, depending on whether or not price indexation is allowed for.[15] These higher values are in accordance with Blinder *et al.* (1998) and Rotemberg and Woodford (1998), but contrast the high degree of price rigidity estimated by SW (2004).

Our estimates of σ are rather large. With no habits, these estimates map directly with the intertemporal elasticity of substitution and suggest that this may be quite small.[16] A common theme in papers estimating DSGE models is the difficulty in pinning down ϕ. Therefore, it is not surprising that, inference on the inverse Frisch elasticity of labor supply is susceptible to the specification of the model, and exhibits wide posterior probability intervals.

Turning to the coefficient governing habit formation, h is tightly estimated and suggests rather inertial consumption and output processes. Reported posterior intervals for h are almost identical to the ones obtained by Juillard *et al.* (2004) and higher than the estimates

[15] Using $\chi \equiv \frac{(1-\beta\xi)(1-\xi)}{(1+\beta\gamma)\xi}$ we obtain $\xi = 0.67, 0.36, 0, 60, 0.53$ corresponding to contract lengths, $\frac{1}{1-\xi}$, of 3.06, 1.57, 2.50 and 2.13 quarters for models G, H, GH and Z respectively.

[16] This result is attributable to a prior density centered on high values for σ. Redoing the estimation using the SW priors leads to point estimates far closer to one, clearly revealing that inference on this parameters is sensitive to the choice of priors.

by SW. By contrast, the posterior density of γ lies to the left of our chosen prior, suggesting, in contrast to studies mentioned earlier, that inflation is intrinsically not very persistent – a result that accords with findings in Erceg and Levin (2001), Taylor (2000) and Cogley and Sargent (2001).

Estimates of the shock processes reveal that both the technology and the government expenditure shock are highly persistent, and this holds true regardless of the exact model specification. Posterior estimates clearly attribute greater volatility of shocks to the government expenditure component rather than to disturbances in technology.[17] As usual, exogenous disturbances to the monetary policy equation appear much less important than technology and government expenditure shocks in driving inflation, consumption and output processes.[18]

2.5.5 Model Comparison

Since the goal of this paper is to characterize the design of robust rules under uncertainty, it is important to investigate which specification seems to be best supported by the data. In doing so we do not intend to select any particular model as being the 'true' one but rather wish to compute posterior probabilities to place odds on the different models.

Bayesian methods for model comparisons allows us to obtain these posterior model probabilities in order to discriminate or aggregate across competing specifications, thereafter providing coefficient estimates that explicitly account for model uncertainty. Let us define m_k to be one possible element from the (discrete) set of competing models $\mu = \{G, H, GH, Z\}$. The posterior model probability for $p(m_k|Y^T)$ summarizes the evidence provided by the data in favor of m_k and is then given by

$$p(m_k|Y^T) = f(Y^T|m_k)p(m_k)/f(Y^T) \tag{27}$$

where $p(m_k)$ stands for the prior probability assigned to model k, that in our case equals $\frac{1}{4}$ since we treat each model as equiprobable a-priori. The first expression in the numerator is known as the marginal likelihood (or marginal data density) and was previously presented

[17] Indeed, the 1st posterior decile of the former exceeds the 9th decile of the latter, for all models, despite similar prior densities for the innovation standard deviations.

[18] Note that the correlation of shocks is important as well. So far, as it is standard in most models, we have constrained the disturbances to be i.i.d.

as the denominator in equation (26) We compute the posterior model probabilities using the Reversible Jump MCMC algorithm (RJMCMC) of Dellaportas *et al.* (2002)). This method belongs to the class of product space search algorithm that adds a model indictor variable to be estimated jointly with the parameters.

Estimates of $p(m_k|Y^T)$ obtained with the RJMCMC for our four model variants are presented in **Table B3**. In line with results discussed above, the specification with habit persistence and no price indexation (G) attains highest posterior probability. Model Z, which allows for both of these intrinsic mechanisms, follows in probability ranking. In contrast, a model with no habit persistence is 9 times less likely than those specifications (Z and G) with endogenous persistence in consumption. Finally, the most restrictive model., GH attains the lowest posterior model probability further providing evidence of the need to incorporate at least one of the two intrinsic mechanisms imparting greater inertia to the model. Therefore, these results can be interpreted as suggesting that the addition of endogenous mechanisms of persistence, particularly habit in consumption, improve the fit of the model. These posterior odds will be used to weight the models for our analysis of uncertainty on the robustness of policy rules.

In our policy analysis we confine ourselves to models G, GH and Z. The reason for choosing GH over H is twofold. First, the degree of price flexibility suggested by the H model (contract length 1.57 quarters) seems implausible. Second, an alternative Modified Harmonic Mean estimator proposed by Geweke (1999) found that the H-model posterior probability was much lower than that of the GH model.

3 The Stability and Determinacy of IFB Rules

3.1 Theory

This section studies an IFB rule of the form

$$\begin{aligned} i_t &= \rho i_{t-1} + \theta(1-\rho)\mathcal{E}_t\pi_{t+j}\,;\ \rho \in [0,1),\, \theta > 0 \\ &= i_{t-1} + \Theta\mathcal{E}_t\pi_{t+j}\,;\ \rho = 1,\ \Theta > 0 \end{aligned} \tag{28}$$

where $j \geq 0$ is the forecast horizon, which is a feedback on single-period inflation over the period $[t+j-1, t+j]$.[19] With rule (28), policymakers set the nominal interest rate so as to respond to deviations of the inflation term from target. In addition, policymakers smooth rates, in line with the idea that central banks adjust the short-term nominal interest rate only partially towards the long-run inflation target, which is set to zero for simplicity in our set-up. The parameter $\rho \in [0, 1]$ measures the degree of interest rate smoothing. If $\rho = 1$ we have an *integral rule* that guarantees that the long-run inflation target (zero in our set-up) is met, provided the rule stabilizes the economy. For $\rho < 1$, (28) can be written as $\Delta i_t = \frac{1-\rho}{\rho}[\theta \mathcal{E}_t \pi_{t+j} - i_t]$ which is a partial adjustment to a static IFB rule $i_t = \theta \mathcal{E}_t \pi_{t+j}$. j is the feedback horizon of the central bank. When $j = 0$, the central bank feeds back from current dated variables only. When $j > 0$, the central bank feeds back instead from deviations of forecasts of variables from target. Finally, $\theta, \Theta > 0$ are the feedback parameters for the non-integral and integral rules respectively: the larger is θ or Θ, the faster is the pace at which the central bank acts to eliminate the gap between expected inflation and its target value.

To understand better how the precise combination of the pairs (j, θ) or (j, Θ) in IFB rules can lead the economy into instability or indeterminacy consider the deterministic model economy (24) and (25) with interest rate rules of the form (28). g_t and a_t are exogenous stable processes and play no part in the stability analysis. For convenience, we therefore set them to zero. Let z be the forward operator. Taking z-transforms of (16), (17), (18) and (28), the characteristic equation for the system is given by:

$$\begin{aligned} & (z-\rho)[(z-1)(z-h)(\beta z-1)(z-\gamma) - \frac{\lambda}{\mu} z^2 (\tilde{\phi} z + \mu(z-h))] \\ + \quad & \frac{\lambda\theta}{\mu}(1-\rho)(\tilde{\phi} z + \mu(z-h)) z^{j+2} = 0 \end{aligned} \tag{29}$$

for non-integral rules and

$$\begin{aligned} & (z-1)[(z-1)(z-h)(\beta z-1)(z-\gamma) - \frac{\lambda}{\mu} z^2 (\tilde{\phi} z + \mu(z-h))] \\ + \quad & \frac{\lambda\Theta}{\mu}(\tilde{\phi} z + \mu(z-h)) z^{j+2} = 0 \end{aligned} \tag{30}$$

[19] To set the model up with this rule in state-space form for $j \leq 1$ we need to augment the state vector with a lagged term i_{t-1}. For $j = 2$ replace t with $t+1$ in (16)-(17) and take expectations at time t. Then the state-space presentations remains of the same dimension. For $j > 2$ replace t with $t+j-1$ in (16)-(17) and take expectations at time t. The state vector must then be augmented with $\mathcal{E}_t \pi_{t+1} \cdots \mathcal{E}_t \pi_{t+j-2}$.

for integral rules. In these characteristic equations we have defined $\lambda \equiv \frac{(1-\beta\xi)(1-\xi)}{\xi}$, $\tilde{\phi} \equiv \frac{\bar{C}}{\bar{Y}}\phi$ and $\mu \equiv \frac{\sigma}{1-h}$. Equations (29) and (30) show that the minimal state-space form of the system has dimension $\max(5, j+3)$. Since there are 3 predetermined variables in the system, it follows that the saddle-path condition for a unique stable rational expectations solution is that the number of roots inside the unit circle of the complex plane is 3 and the number outside the unit circle is $\max(2, j)$.

In the analysis that follows we focus on integral rules with characteristic equation (30).[20] To identify values of (j, Θ) that involve exactly three roots of equation (30) we graph the root locus of (Θ, z) pairs that traces how the roots change as Θ varies between 0 and ∞. All the graphs can be drawn by following the rules set out in Appendix A of BLP. Other parameters in the system, including the feedback horizon parameter j in the IFB rule, are kept constant. We generate separate charts, each conditioning on a different horizon assumption. Each chart shows the complex plane (indicated by the solid thin line),[21] the unit circle (indicated by the dashed line), and the root locus tracking zeroes of equation (30) as Θ varies between 0 and ∞ (indicated by the solid bold line). The arrows indicate the direction of the arms of the root locus as Θ increases. Throughout we experiment with both a 'high' and a 'low' $\frac{\lambda}{\mu}$, as defined after (29). The economic interpretation of these cases is that the high $\frac{\lambda}{\mu}$ case corresponds to low ξ (i.e., more flexible prices) and low $\frac{\sigma}{1-h}$ (low risk aversion and habit formation).

The term inside the square brackets in equation (30) corresponds to no nominal interest rate feedback rule (i.e., an open-loop interest rate policy). Then rule (28) is switched off and so the lagged term i_{t-1} disappears from our model; the system now requires exactly *two* stable roots for determinacy. Figure 1 plots the root locus which, in this case, is just a set of dots: namely, the roots of the polynomial in the square brackets of (30). Note that depending on the value of λ/μ, the position of these roots varies, and in the flexible price, low interdependence case where $\frac{\lambda}{\mu}$ is high, there are complex roots indicating oscillatory dynamics. The diagram shows that there are *too many* stable roots in both cases (i.e. 3 instead of 2), which implies that with no interest rate feedback rule, there will always be

[20] The corresponding analysis for non-integral rules is to be found in BLP.

[21] In this plane, the horizontal axis depicts real numbers, and the vertical axis depicts imaginary numbers. If a root is complex, i.e. $z = x + iy$, then its complex conjugate $x - iy$ is also a root. Thus the root locus is symmetric about the real axis.

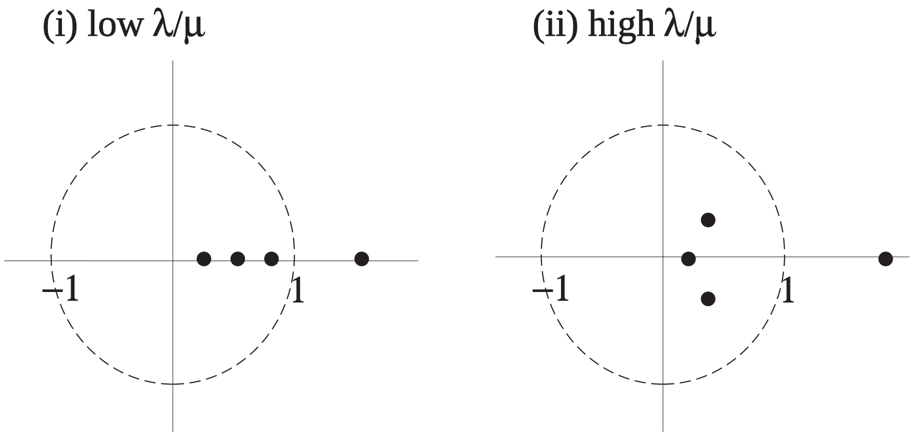

Figure 1: **Position of zeroes with no Interest Feedback Rule.**

indeterminacy in the system.

If the nominal interest rate rule is switched on and now feeds back on current rather than expected inflation, i.e. $j = 0$, then the root locus technique yields a pattern of zeroes as depicted in Figure 2. Integral control brings about a lag in the short-term nominal interest rate and the system is now stable if it has exactly *three* stable roots (as we now have three predetermined variables in the system). The figure demonstrates that if $\Theta > 0$ one arm of the root locus starting originally at $z = 1$ exits the unit circle, turning one root from unity to unstable so that there are now three – as required – instead of four stable roots and the system has a determinate equilibrium. As $\Theta \to \infty$, there are roots at $\pm i\infty$, two roots at 0, and one at $\mu h/(\tilde{\phi} + \mu)$, the latter shown as a square.

Thus we conclude that for a rule feeding back on current inflation, the system exhibits determinacy if and only if $\Theta > 0$. For higher values of $j \geq 1$ we can draw the sequence of root locus diagrams shown in Figures 3 and 4. Our diagrams show that an arm of the root locus re-enters the unit circle for some high $\Theta > 1$ and indeterminacy re-emerges. Therefore $\Theta > 0$ is necessary but not sufficient for stability and determinacy. Our results up to this point are summarized in proposition 1:

Proposition 1**: For an integral rule feeding back on current inflation ($j = 0$), $\Theta > 0$ is a necessary and sufficient condition for stability and determinacy. For higher feedback horizons ($j \geq 1$), $\Theta > 0$ is a necessary but not sufficient condition for stability and determinacy.**

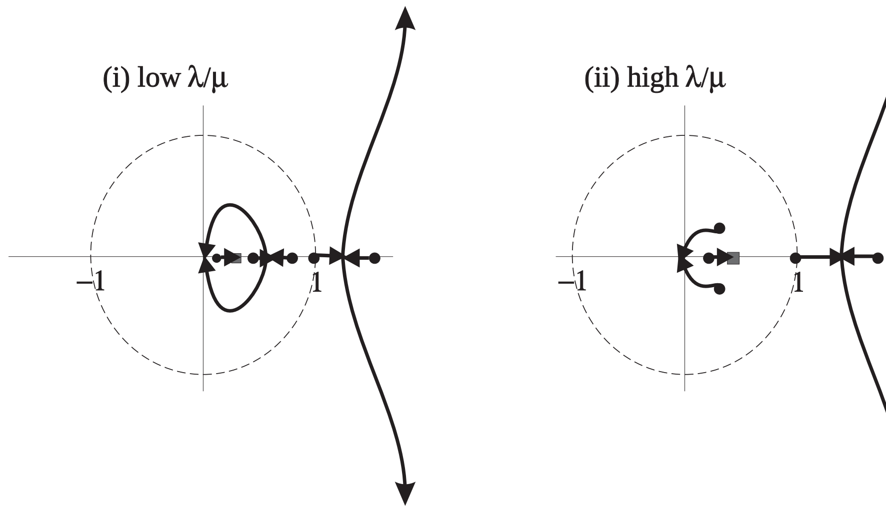

Figure 2: **Integral Control IFB Rule on Current Inflation: Position of Zeroes as Θ Changes.**

Now let $\bar{\Theta}(j)$ be the upper critical value of Θ for the system for a feedback horizon j. Figure 3 shows that for the case $j = 1$ indeterminacy occurs when this portion of the root locus enters the unit circle at $z = -1$. The critical upper value for $\Theta = \bar{\Theta}(1)$ when this occurs is obtained by substituting $z = -1$ and $j = 1$ into the characteristic equation (30) to obtain:

$$\bar{\Theta}(1) = 2\left[1 + \frac{2(1+h)(1+\beta)(1+\gamma)\mu}{\lambda(\tilde{\phi} + \mu(1+h))}\right] \quad (31)$$

One important thing to note looking at this expression for a 1-period ahead IFB integral rule is that $\bar{\Theta} > 2$ and that the problem of indeterminacy lessens for high h, γ and σ, and low λ and $\tilde{\phi}$. Notice from the definition of λ after (30) that low λ is associated with a high degree of price stickiness.

Proceeding on to j-period ahead IFB rules, for $j \geq 2$ the analysis is more difficult. For $j = 2$, Figure 4 shows that indeterminacy occurs when the root locus enters the unit circle at $z = cos(\psi) + isin(\psi)$ for some $\psi \in (0, \frac{\pi}{2})$. A similar reasoning applies to $j > 2$. All our results up to this point are analytical using topological reasoning, but now the threshold $\bar{\Theta}(j)$ for $j \geq 2$ must be found numerically. Given j, write the characteristic equation as

$$\sum_{k=1}^{max(5,j+3)} a_k(\Theta)z^k = 0 \quad (32)$$

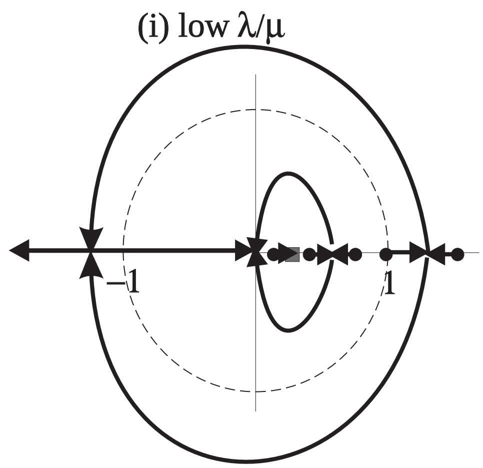

Figure 3: **Integral Control IFB Rule on 1-Period Ahead Expected inflation: Position of Zeroes as Θ Changes.**

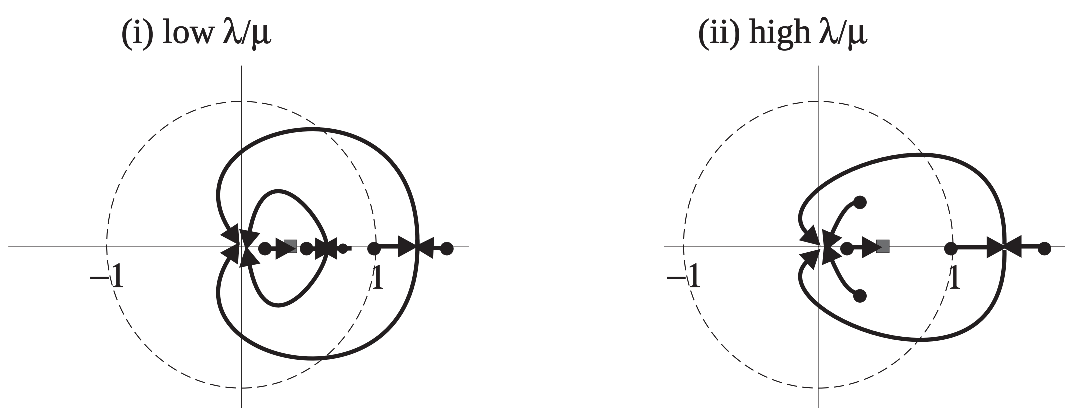

Figure 4: **Integral Control IFB Rule on 2-Period Ahead Expected inflation: Position of Zeroes as Θ Changes.**

noting that some of the a_k are dependent on Θ. The root locus meets the unit circle at $z = cos(\psi) + isin(\psi)$. Using De Moivre's theorem $z^k = cos(k\psi) + isin(k\psi)$ and equating real and imaginary parts we arrive at two equations which can be solved numerically for $\bar{\Theta}$ and ψ. We can summarize these analytical results as:

Proposition 2: For j-period ahead integral IFB rules, $j \geq 1$, there exists a range $\Theta \in [0, \bar{\Theta}(j)]$ with $\bar{\Theta}(j) > 0$ such that the model is stable and determinate.

These results for integral IFB rules contrast with those for non-integral rules (29) studied in BLP. There we found that proposition 1 is modified to a generalized 'Taylor principle': $\theta > 1$ is necessary and sufficient for a $j = 1$ IFB non-integral rule, but only necessary for $j > 1$. For non-integral rules as j increases a more interesting result emerges, namely there is always some lead J such that for

$$j > J = \frac{1}{1-\rho} + \frac{(1-\beta)(1-\gamma)\sigma}{\lambda(\tilde{\phi}+\sigma)} \tag{33}$$

there is indeterminacy *for all values of* θ.[22] This comparison between integral and non-integral rules shows the benefit of the former in avoiding the problem of indeterminacy. Then there is always at least a 'minimal control' with the feedback coefficient Θ very close to zero that is sufficient to achieve stability and determinacy.[23] However with non-integral rules for sufficiently high $j > J$ no such IFB rule is available.

To get a feel for these results we provide numerical results for threshold values $\bar{\theta}$ for non-integral rules and $\bar{\Theta}$ for integral rules. Tables 1a-1c set parameter values at their median values for models G, GH and Z respectively. For non-integral rules we set $\rho = 0.8$.

Threshold	ρ	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
$\bar{\theta}(j)$	0.8	222	24	5.5	1.8	1.3	indeterminacy
$\bar{\Theta}(j)$	1	51	7.4	2.2	1.0	0.62	0.43

Table 1a. Critical upper bounds for $\bar{\theta}(j)$ and $\bar{\Theta}(j)$ for Model G.

[22] There are some conditions for this result to hold discussed in BLP. Numerical results indicate these conditions hold for all realistic values of the parameters, and certainly those estimated in section 2.

[23] The utility outcome of a j-period ahead minimal control for high j may however be very poor, as we shall see later.

Threshold	ρ	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
$\bar{\theta}(j)$	0.8	102	12	3.4	1.7	1.0	indeterminacy
$\bar{\Theta}(j)$	1	23	3.6	1.2	0.68	0.46	0.34

Table 1b. Critical upper bounds for $\bar{\theta}(j)$ and $\bar{\Theta}(j)$ for Model GH.

Threshold	ρ	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
$\bar{\theta}(j)$	0.8	119	19	3.2	1.6	1.00	indeterminacy
$\bar{\Theta}(j)$	1	23	5.1	1.2	0.57	0.40	0.31

Table 1c. Critical upper bounds for $\bar{\theta}(j)$ and $\bar{\Theta}(j)$ for Model Z.

These numerical results confirm the predictions of our theory. For each model the indeterminacy problem becomes more acute as the horizon j increases imposing a tighter constraint on the range of IFB rules available. For non-integral rules with $\rho = 0.8$, the maximum horizon J is just over 5 quarters as predicted by (33). In accordance with proposition 2, for integral rules as j increases there is always some feedback coefficient on expected inflation $0 < \Theta < \bar{\Theta}$ such that the IFB rule yields stability and determinacy. For $j \geq 3$, model Z with inflation and output persistence is most prone to indeterminacy.

3.2 Weakly Robust IFB Rules

We are now in a position to identify weakly robust rules; i.e., those IFB rules that guarantee stability and determinacy. Weakly M-robust rules give stability and determinacy across parameter specifications corresponding to median values in models G, GH and Z. Weakly P-robust rules give stability and determinacy for all possible parameter specifications across a large number of draws.

Consider first non-integral rules. Regions to the south-west of each contour corresponding to a choice of IFB horizon j in Figure 5 shows the regions for parameters ρ and θ that yield weakly M-robust rules. Figure 6 is based on 10000 draws of parameter combinations across all possible models using the estimated posterior parameter distributions of section 2.5. Regions to the south-west of each contour then represent 100% confidence regions of determinacy for this sample and give rules that are weakly P-robust. For both M-robustness and P-robustness, the declining size of this region as the forward horizon

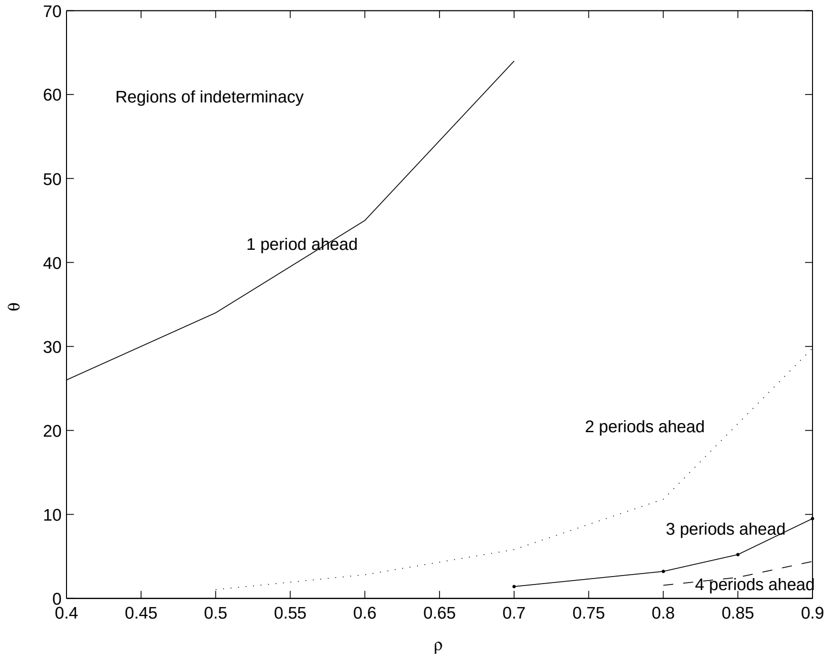

Figure 5: **Non-integral Rules: Regions of Weak M-Robustness.**

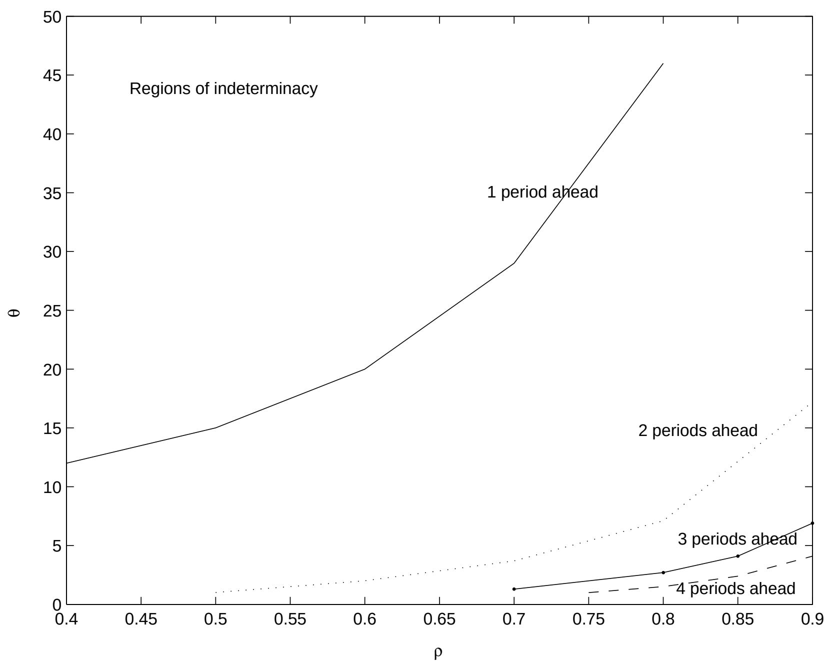

Figure 6: **Non-integral Rules: Regions of Weak P-Robustness.**

j increases confirms the earlier theoretical results that show that IFB rules with unique and stable equilibria are increasingly constrained in the choice of (ρ, θ) with a qualitative change taking place between $j = 1$ and $j = 2$.

Consider next integral rules. Weakly M-robust rules can be identified from tables 1a-1c by picking out the minimal thresholds $\bar{\Theta}(j)$ across the three models. For weakly P-robust we pick out the minimal thresholds based on 10000 draws of parameter combinations as before. The results in table 2 confirm that the requirement of M-robustness, and especially P-robustness, increasingly constrain the IFB rule as j increases.

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
M-Robustness: $\bar{\Theta}(j)$	23	3.6	1.2	0.57	0.40	0.32
P-Robustness: $\bar{\Theta}(j)$	7.0	2.0	0.37	0.36	0.30	0.25

Table 2. Integral Rules: Critical upper bounds for $\bar{\Theta}(j)$.

4 Optimal Policy and Optimized IFB Rules without Model Uncertainty

Without model uncertainty, the policy problem of the central bank at time $t = 1$ is to choose in each period $t = 1, 2, 3, \cdots$ an interest rate i_t so as to minimize a standard expected loss function that depends on the variation of the output relative to an an output target $o_t = y_{nt} + k$, inflation and the change in the nominal interest rate[24]:

$$\Omega_0 = \mathcal{E}_0 \left[\frac{1}{2} \sum_{t=0}^{\infty} \beta_c^t \left[(y_t - o_t)^2 + b\pi_t^2 + c(i_t - i_{t-1})^2 \right] \right] \tag{34}$$

where β_c is the discount factor of the central bank. The term k is ambitious target for output that exceeds the natural level of output. It arises because the natural level of output is not efficient (owing to mark-up pricing in a monopolistically competitive intermediate goods sector, market power in the labour market and habit persistence).

[24]Notice this is a central bankers' loss function, not a welfare function. It describes the actual policy objectives banks have (or are instructed to have) rather than what they should have.

4.1 Optimal Policy with and without Commitment

We first compute the optimal policies where the policy maker can commit, and the optimal discretionary policy where no commitment mechanism is in place.[25] In our linear-quadratic framework optimal policies (including those for optimal IFB rules) conveniently decompose into deterministic and stochastic components. Let target variables in (34) be written as sums of a deterministic stochastic components such as $y_t = \bar{y}_t + \tilde{y}_t$ where all variables are expressed in deviation form about the baseline zero-inflation deterministic steady state of the known model. Then the expected loss function decomposes as

$$\begin{aligned}\Omega_0 &= \frac{1}{2}\sum_{t=0}^{\infty}\beta_c^t\left[(\bar{y}_t - \bar{o}_t)^2 + b\bar{\pi}_t^2 + c(\bar{i}_t - \bar{i}_{t-1})^2 + \mathcal{E}_0\left[(\tilde{y}_t - \tilde{o}_t)^2 + b\tilde{\pi}_t^2 + c(\tilde{i}_t - \tilde{i}_{t-1})^2\right]\right] \\ &= \bar{\Omega}_0 + \tilde{\Omega}_0 \qquad (35)\end{aligned}$$

say. The policymaker can then design an optimal policy consisting of an open-loop trajectory that minimizes $\bar{\Omega}_0$ subject to the deterministic model plus a feedback rule that minimizes $\tilde{\Omega}_0$ subject to a stochastic model expressing stochastic deviations about the open-loop trajectory. By the property of *certainty equivalence* for full optimal policies, but *not* optimized simple rules, the feedback rule is independent of both the initial values of the predetermined variables and the variance-covariance matrix of the disturbances.

The optimal policy under commitment provides a benchmark with which to compare the loss in other policy rules. We use the optimal discretionary policy with $k = 5\%$ to calibrate b to result in an annual inflationary bias (the long-run inflation rate) of 5%. This gave $b = 2.5, 1.5, 0.85$ for models G, GH and Z respectively. The discount factor of the central bank was set at $\beta_c = 0.988$ which corresponds to an annual discount rate of 5%. We then set the weight c just sufficiently high to avoid a negative interest rate anywhere along the trajectory of the optimal policies. This required $c_G = c_{GH} = 3$, $c_Z = 2$. Figures 7 and 8 show the deterministic component of inflation and the nominal interest rate for model G and demonstrate that these calibration requirements are met for b_G and c_G.

[25] Full details of the procedures used to compute optimal policies and optimized IFB rules are provided in Appendix B.

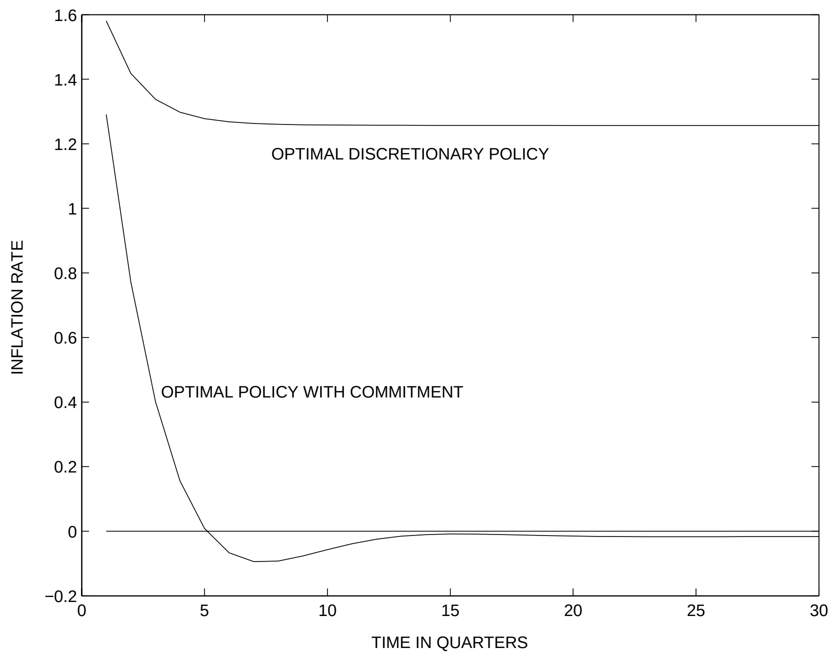

Figure 7: **The Quarterly % Inflation Rate under the Deterministic Optimal Policy for model G.** $k = 5\%$**.**

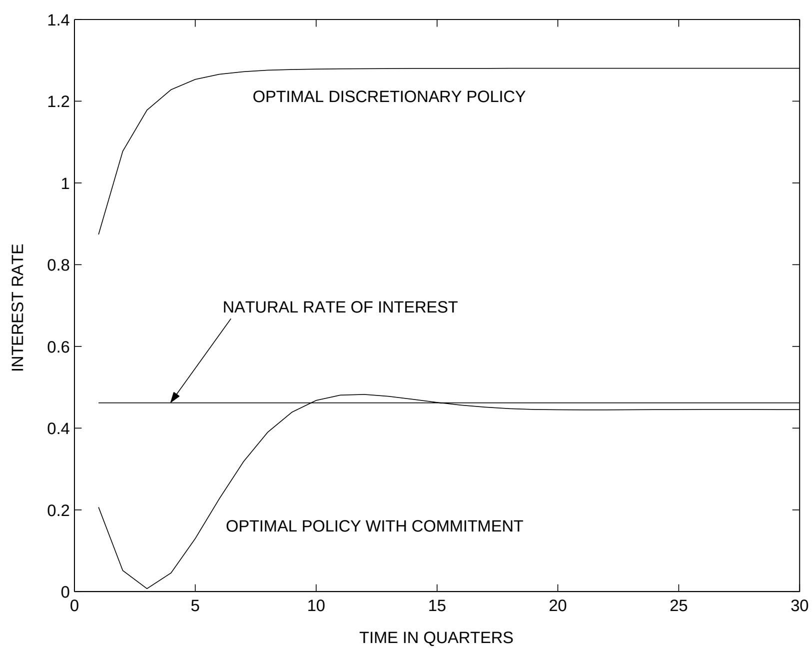

Figure 8: **The The Quarterly % Interest Rate under the Deterministic Optimal Policy for model G.** $k = 5\%$**.**

4.2 Optimized IFB Rules

We now turn to optimized IFB rules and optimal Taylor-type rules feeding back on either current inflation alone or on inflation and the output gap. The general form of the rule that covers integral and non-integral IFB as well as the Taylor-type rules is given by

$$i_t = \rho i_{t-1} + \Theta \mathcal{E}_t \pi_{t+j} + \Theta_y (y_t - y_{nt})\,; \;\; \rho \in [0,1],\, \Theta, \Theta_y > 0,\, j \geq 0 \tag{36}$$

In all the results from this point onwards we focus exclusively on stabilization policy by putting $k = 0$ so there is no deterministic component of policy in response to an ambitious output target.[26] Given the estimated variance-covariance matrix of the white noise disturbances, an optimal combination (Θ, ρ) can be found for each rule defined by the time horizon $j \geq 0$, and for the Taylor rule, and optimal combination (Θ, Θ_y, ρ). The results are shown in tables 3 to 6 for the estimated models G, GH and Z of section 2.5. The Taylor rule is for $j = 0$ only.

A number of interesting observations emerge from these tables. First, from the output equivalent loss (relative to the optimal commitment outcome) of 'minimal control', the closest saddle-path stable rule using current inflation to no feedback rule at all, we see that there are significant though not dramatic gains from stabilization of between 0.5%-1.0% across the three models. Second, the output equivalent loss from optimal discretion indicate only small stabilization gains from commitment if the latter policy rule is implementable. By far the main gain from commitment is the elimination of the inflationary bias which has been ruled out in these results by putting $k = 0$. Third, if the policymaker can commit using a simple rule, the best one in this respect is a Taylor integral rule, and this realizes a large part of the potential stabilization gain. Third, for each model we search for optimized rules within those that satisfy the determinacy conditions on ρ and θ for non-integral rules and on Θ for integral rules. Our theory has shown that this requirement severely constrains the range of possible stabilizing rules as the horizon j increases and as a result compared with the Taylor rule, IFBj rules *perform increasingly less well*. In our results the transition from IFB3 to IFB4 is particularly dramatic involving an output equivalent loss of between 1% (for model G) and 55% (model Z). This is what

[26] Since the IFB rule assumes a commitment mechanism, the policymaker in principle should be able to implement a policy $i_t = \bar{i}_t$ plus a feedback component such as (28) or (36) relative to $\bar{i}_t$, where the latter is the optimal deterministic trajectory found in the previous section.

our theory leads us to expect from table 1 since the determinacy requirement imposes the tightest constraint on model Z.

Rule	ρ	Θ	Θ_y	Loss Function	% Output Equivalent
Minimal Feedback on π_t	1	0.001	0	39.1	0.93
IFB0	1	1.43	0	2.27	0.03
Taylor Rule	1	0.81	1.00	1.86	0.02
IFB1	1	3.99	0	2.63	0.04
IFB2	1	5.00	0	3.19	0.05
IFB3	1	2.17	0	9.78	0.21
IFB4	1	1.02	0	44.0	1.05
Optimal Commitment	n.a.	n.a.	n.a	1.07	0
Optimal Discretion	n.a.	n.a.	n.a	2.98	0.05

Table 3. Model G: Optimal Rules and Optimized Simple Rules Compared.[27]

Rule	ρ	Θ	Θ_y	Loss Function	% Output Equivalent
Minimal Control π_t	1	0.001	0	30.3	0.72
IFB0	1	1.40	0	1.44	0.02
Taylor Rule	1	0.77	1.00	1.37	0.02
IFB1	1	5.00	0	1.61	0.02
IFB2	1	3.59	0	2.70	0.05
IFB3	1	1.23	0	21.3	0.50
IFB4	1	0.66	0	147	3.56
Optimal Commitment	n.a.	n.a.	n.a	0.64	0
Optimal Discretion	n.a.	n.a.	n.a	1.78	0.03

Table 4. Model GH: Optimal Rules and Optimized Simple Rules Compared.

[27]IFBj denotes a j-period ahead IFB rule. Let Ω=loss from rule, Ω^O =loss from optimal rule with commitment. A 1% permanent fall in the output gap leads to a reduction in the loss function of $\frac{1}{2(1-\beta_c)} = 41$ in our calibration. The % output equivalent loss is then a measure of the degree of sub-optimality of each Rule and is defined as $\frac{\Omega - \Omega^O}{41} \times 100$. Optimized simple rules were restricted to the ranges $\rho \in [0,1]$ and $\Theta \in [1,5]$.

Rule	ρ	Θ	Θ_y	Loss Function	% Output Equivalent
Minimal Feedback on π_t	1	0.001	0	22.45	0.54
IFB0	1	1.25	0	0.88	0.01
Taylor Rule	1	1.25	0.11	0.88	0.01
IFB1	1	2.96	0	1.16	0.02
IFB2	1	5.0	0	1.45	0.02
IFB3	1	1.17	0	28.03	0.67
IFB4	0.87	0.40	0	2314	54.8
Optimal Commitment	n.a.	n.a.	n.a	0.45	0
Optimal Discretion	n.a.	n.a.	n.a	0.93	0.01

Table 5. Model Z: Optimal Rules and Optimized Simple Rules Compared.

5 Robust IFB Rules with Model Uncertainty

5.1 Theory

In this section we consider model uncertainty in the form of uncertain estimates of the non-policy parameters of the model, $\Theta = (\beta, \gamma, \xi, \phi, \sigma, h, \rho_a, \rho_b, \zeta, \eta, \kappa, \sigma_{at}^2, \sigma_{gt}^2)$. Suppose the state of the world s is described by a model with $\Theta = \Theta^s$ expressed in state-space form as

$$\begin{bmatrix} \mathsf{z}_{t+1}^s \\ \mathcal{E}_t \mathsf{x}_{t+1}^s \end{bmatrix} = A^s \begin{bmatrix} \mathsf{z}_t^s \\ \mathsf{x}_t^s \end{bmatrix} + B^s i_t^s + C^s \begin{bmatrix} \epsilon_{gt+1} \\ \epsilon_{at+1} \end{bmatrix} \tag{37}$$

$$o_i^s = E^s \begin{bmatrix} \mathsf{z}_t^s \\ \mathsf{x}_t^s \end{bmatrix} \tag{38}$$

where $\mathsf{z}_{t-1}^s = [a_t^s, g_t^s, c_{t-1}^s, c_{n,t-1}^s, \pi_{t-1}^s]$ is a vector of predetermined variables at time t and $\mathsf{x}_t = [c_t^s, \pi_t^s]$ are non-predetermined variables in state s of the world. In (37) and (38) it is important to stress that variables are in deviation form about a zero-inflation steady state of the model in state s. For example output in deviation form is given by $y_t^s = \frac{Y_t^s - \bar{Y}^s}{\bar{Y}_s}$ where $\bar{Y}^s$ is the steady state of the model in state s defined by parameters Θ^s and $i_t^s = i_t - \bar{i}^s$ where the natural rate of interest in model s, $\bar{i}_s = \frac{1}{\beta^s} - 1$.

For M-robustness, in general one sets up a *composite model* of outputs from each of the states $s = 1, 2, \cdots, n$ corresponding to the rival models and minimizes the expected loss across these states using estimated posterior probabilities. Because each model is linearized about a different steady state, we must now set up the model in state s in terms of the *actual* interest rate, not the deviation about the steady state. Then augmenting the state vector to become $\mathbf{z}_t^s = [1, a_t^s, g_t^s, c_{t-1}^s, c_{n,t-1}^s, \pi_{t-1}^s]$ we still have a state have a state-space form (37) and (38) and we minimize

$$\begin{aligned}\Omega_0 &= \frac{1}{2}\sum_{t=0}^{\infty}\beta_c^t \sum_{s=1}^{n} p_s\Big[(\bar{y}_t^s - \bar{o}_t^s)^2 + b_s(\bar{\pi}_t^s)^2 + c_s(\bar{i}_t - \bar{i}_{t-1})^2 \\ &+ \mathcal{E}_0\left[(\tilde{y}_t^s - \tilde{o}_t^s)^2 + b_s(\tilde{\pi}_t^s)^2 + c_s(\tilde{i}_t - \tilde{i}_{t-1})^2\right]\Big] \end{aligned} \tag{39}$$

For P-robustness (39) is replaced with the average expected utility loss across a a large number of draws from all models constructed using both the posterior model probabilities and the posterior parameter distributions for each model.

In (39) the output target in state s of the world is given by $o_t^s = y_{nt}^s + k^s$ where the ambitious output target k^s depends on s. In fact we will continue to assume that the central bank has no ambitious output targets and set $k^s = 0$ in its loss function. However with model uncertainty there is still a deterministic component of policy arising from differences in the natural rate of interest compatible with zero inflation in the steady state, $\bar{i}^s = \frac{1}{\beta^s} - 1$.[28] A non-integral rule specifying $i_t = \bar{i}^s$ in the long-run will only result in zero inflation in model s. From the consumers' Euler equation (4) in model r with $\beta^r > \beta^s$, implementing the rule designed for model s with $\bar{i} = \bar{i}^s = \frac{1}{\beta^s} - 1$ gives a steady state inflation rate $\bar{\pi}^r$ that is no longer zero but given by

$$\frac{\beta^r(1+\bar{i}^s)}{(1+\bar{\pi}^r)} = \frac{\beta^r}{\beta^s(1+\bar{\pi}^r)} = 1 \quad \text{i.e., } \bar{\pi}^r = \frac{\beta^r}{\beta^s} - 1 > 0 \tag{40}$$

Our robust non-integral rule designed for any model specifies a natural zero inflation rate of interest $\bar{i}_R$, corresponding to a discount factor $\beta_R = \frac{1}{1+\bar{i}_R}$ to result in an expected long-run inflation rate across models of zero. This implies β_R is determined by

$$\sum_{s=1}^{n} p_s\left[\frac{\beta_s}{\beta_R} - 1\right] \Rightarrow \beta_R = \sum_{s=1}^{n} p_s\beta_s \tag{41}$$

[28] In fact estimated differences in β^s between models $s = G, GH, Z$ are not great, so the point we make here is only potentially important.

That is, β_R is the expected value of β_s across the model variants. The need to specify a natural rate of interest, $\bar{i}_R$, only applies to non-integral rules. By contrast, a further benefit of integral rules is that the economy is automatically driven to a zero-inflation steady state whatever the state of the world without having to specify $\bar{i}_R$.

There is one final consideration first raised by Levine (1986) that is usually ignored in the literature. Up to now we have assumed that private sector expectations $\mathcal{E}_t \mathrm{x}^s_{t+1}$ are state s model-consistent expectations. In other worlds in each state of the world the private sector knows the state and faces no model uncertainty. In a more general formulation of the problem we can relax this assumption and assume that both the policymaker and the private sector faces model uncertainty. Suppose that in state s of the world the latter believes model s' with probability $q_{ss'}$. Then $\mathcal{E}_t \mathrm{x}^s_{t+1}$ must be replaced by the composite expectation $\sum_{s'=1}^{n} q_{ss'} \mathcal{E}_t \mathrm{x}^{s'}_{t+1}$ and the composite model no longer decomposes into independent systems. In the results that follow we bypass this complication and confine ourselves to model-consistent expectations in each state of the world.

5.2 Strongly Robust IFB Rules

Rule	ρ	Θ
M-Robust IFB0	1	1.40
P-Robust IFB0	1	1.61
M-Robust IFB1	1	3.93
P-Robust IFB1	1	4.37
M-Robust IFB2	1	3.59
P-Robust IFB2	1	1.85
M-Robust IFB3	1	1.17
P-Robust IFB3	1	0.24
M-Robust IFB4	0.92	0.41
P-Robust IFB4	1	0.24

Table 6. Strongly Robust IFB Rules.

We now present the strongly M-robust and P-robust IFB rules with horizon $j = 0, \cdots, 4$. Table 6 reports these rules, which turn out to be of the integral type in almost all cases, as our theory leads us to expect.

Rule	Model G	Model GH	Model Z
IFB0(G)	2.27 (0.03)	1.44 (0.02)	0.88 (0.01)
IFB0(GH)	2.27 (0.03)	1.44 (0.02)	0.88 (0.01)
IFB0(Z)	2.29 (0.03)	1.45 (0.02)	0.88 (0.01)
IFB0(M-Robust)	2.27 (0.03)	1.44 (0.02)	0.88 (0.01)
IFB1(P-Robust)	2.28 (0.03)	1.45 (0.02)	0.90 (0.01)
IFB1(G)	2.63 (0.04)	1.63 (0.02)	1.18 (0.02)
IFB1(GH)	2.66 (0.04)	1.61 (0.02)	1.18 (0.02)
IFB1(Z)	2.69 (0.04)	1.72 (0.03)	1.16 (0.02)
IFB1(M-Robust)	2.63 (0.04)	1.63 (0.02)	1.18 (0.02)
IFB1(P-Robust)	2.63 (0.04)	1.62 (0.02)	1.19 (0.02)
IFB2(G)	3.19 (0.05)	indeterminacy	1.45 (0.02)
IFB2(GH)	3.66 (0.06)	2.70 (0.05)	1.61 (0.03)
IFB2(Z)	3.19 (0.05)	indeterminacy	1.45 (0.02)
IFB2(M-Robust)	3.66 (0.06)	2.70 (0.05)	1.61 (0.03)
IFB2(P-Robust)	5.70 (0.11)	4.76 (0.10)	2.48 (0.05)
IFB3(G)	9.78 (0.21)	indeterminacy	indeterminacy
IFB3(GH)	17.3 (0.40)	21.3 (0.51)	indeterminacy
IFB3(Z)	18.2 (0.42)	22.8 (0.54)	28.0 (0.67)
IFB3(M-Robust)	18.2 (0.42)	22.8 (0.54)	28.0 (0.67)
IFB3(P-Robust)	95.8 (2.31)	169 (4.12)	674 (16.4)
IFB4(G)	44.0 (1.05)	indeterminacy	indeterminacy
IFB4(GH)	72.3 (1.74)	147 (3.57)	indeterminacy
IFB4(Z)	190 (4.61)	424 (10.3)	2246 (54.8)
IFB4(M-Robust)	154 (3.73)	345 (8.40)	2659 (64.8)
IFB4(P-Robust)	216 (5.24)	528 (12.9)	4536 (111)

Table 7. Value of Loss Function for Different Rules with Model Uncertainty[29]

The diagonal elements of table 7 gives the policymaker's losses obtained previously in tables 3 to 5 when the optimized rule designed for model s=G, GH, Z is implemented in that model. Figures in brackets refer to output equivalent % losses. We refer to these rules as IFBj(s) for horizon j. The off-diagonal entries show the loss outcome when the

[29]IFBj(s) denotes the outcome from the j-horizon IFB rule designed for model s. Each row then gives the value of the loss function for models $s = G, GH, Z$. M-robust rules use the posterior model probabilities $p_G = 0.56$, $p_{GH} = 0.12$ and $p_Z = 0.32$. The % output equivalent losses are in brackets. Diagonal elements correspond to losses in tables 2 to 5.

rule designed for model s is implemented on model $r \neq s$. A striking pattern emerges from this table: whereas the current inflation rule IFB0 and the IFB1 rule are remarkably robust across models, this is no longer true for IFBj for $j \geq 2$. IFB0 and IFB1 rules designed for the wrong model perform well in terms of their stabilization properties and the requirements of M-robustness or even P-robustness have little impact on policy design.

For IFBj rules with $j \geq 2$ optimized rules designed for the wrong model can lead to indeterminacy. M-robust and P-robust rules avoid this indeterminacy by design.[30] However robustness comes at a cost especially as the horizon j goes beyond $j = 2$. For $j = 3$, M-robustness imposes output equivalent costs of between $0.42 - 0.67\%$ and for $j = 4$ between $4 - 65\%$. The most stringent robustness criterion, P-robustness, comes at a cost of over 100% output equivalent if an IFB4 rule is pursued and the world turns out to be correctly represented by model Z.

6 Conclusions

Both our theoretical results on IFB rules in section 3 and our numerical results of that and later sections indicate that they become increasing prone to the problem of indeterminacy as the forward horizon increases from $j = 2$ to $j = 4$. As a consequence the stabilization performance of optimized rules of this type worsens too. M-robust and P-robust rules avoid indeterminacy in an uncertain environment, but at an increasing utility loss as rules become more forward-looking.

In view of these results the question arises: why do central banks pursue forward-looking targeting rules in the first place? Two main reasons for favouring such rules are commonly cited. First, the delayed response of inflation to interest rate changes obliges monetary authorities to react in a pre-emptive fashion to expected inflation in the future. Second, by targeting inflation in the future in a simple and accountable fashion, the central bank can respond to shocks whilst at the same time providing the private sector with assurances that inflation will eventually return to its long-run target of zero inflation, in our set-up. Of these two reasons only the second makes any sense in terms of our analysis. Central banks can only target *forecasts* of future inflation and these can only

[30] In our computations a very large loss utility loss is assigned to rules that lead to instability or indeterminacy.

be conditional on information available at the time the interest rate is set, i.e., the state vector at time t in (24). By committing to a rule that feeds back on inflation $j \geq 1$ periods ahead, since this forecast can be expressed as a linear combination of these state variables, the authority is severely constraining how the interest rate should in effect respond to this information, and it is this constraint that lies at the heart of the poor performance of these rules. It may well be the case that a long forward horizon is necessary to establish the commitment to a low inflation target, but there is clearly a need for this credibility argument to be formalized.

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A Computation of Policy Rules

The general model in deterministic form takes the form

$$\begin{bmatrix} z_{t+1} \\ x^e_{t+1,t} \end{bmatrix} = A \begin{bmatrix} z_t \\ x_t \end{bmatrix} + Bw_t \tag{A.1}$$

where z_t is an $(n-m)\times 1$ vector of predetermined variables including non-stationary processed, z_0 is given, w_t is a vector of policy variables, x_t is an $m \times 1$ vector of non-predetermined variables and $x^e_{t+1,t}$ denotes rational (model consistent) expectations of x_{t+1} formed at time t. Then $x^e_{t+1,t} = x_{t+1}$ and letting $y_t^T = \begin{bmatrix} z_t^T & x_t^T \end{bmatrix}$ (A.1) becomes

$$y_{t+1} = Ay_t + Bw_t \tag{A.2}$$

Define target variables s_t by

$$s_t = My_t + Hw_t \tag{A.3}$$

and the policy-maker's loss function at time t by

$$\Omega_t = \frac{1}{2}\sum_{i=0}^{\infty} \lambda^t [s^T_{t+i} Q_1 s_{t+i} + w^T_{t+i} Q_2 w_{t+i}] \tag{A.4}$$

which we rewrite as

$$\Omega_t = \frac{1}{2}\sum_{i=0}^{\infty} \lambda^t [y^T_{t+i} Q y_{t+i} Q y_{t+i} + 2y^T_{t+i} U w_{t+i} + w^T_{t+i} R w_{t+i}] \tag{A.5}$$

where $Q = M^T Q_1 M$, $U = M^T Q_1 H$, $R = Q_2 + H^T Q_1 H$, Q_1 and Q_2 are symmetric and non-negative definite R is required to be positive definite and $\lambda \in (0,1)$ is discount factor. The procedures for evaluating the three policy rules are outlined in the rest of this appendix (or Currie and Levine (1993) for a more detailed treatment).

A.1 The Optimal Policy with Commitment

Consider the policy-maker's *ex-ante* optimal policy at $t = 0$. This is found by minimizing Ω_0 given by (A.5) subject to (A.2) and (A.3) and given z_0. We proceed by defining the Hamiltonian

$$H_t(y_t, y_{t+1}, \mu_{t+1}) = \frac{1}{2}\lambda^t (y_t^T Q y_t + 2y_t^T U w_t + w_t^T R w_t) + \mu_{t+1}(Ay_t + Bw_t - y_{t+1}) \tag{A.6}$$

where μ_t is a row vector of costate variables. By standard Lagrange multiplier theory we minimize

$$L_0(y_0, y_1, \ldots, w_0, w_1, \ldots, \mu_1, \mu_2, \ldots) = \sum_{t=0}^{\infty} H_t \tag{A.7}$$

with respect to the arguments of L_0 (except z_0 which is given). Then at the optimum, $L_0 = \Omega_0$.

Redefining a new costate vector $p_t = \lambda^{-1}\mu_t^T$, the first-order conditions lead to

$$w_t = -R^{-1}(\lambda B^T p_{t+1} + U^T y_t) \tag{A.8}$$

$$\lambda A^T p_{t+1} - p_t = -(Qy_t + Uw_t) \tag{A.9}$$

Substituting (A.8) into (A.2)) we arrive at the following system under control

$$\begin{bmatrix} I & \lambda BR^{-1}B^T \\ 0 & \lambda(A^T - UR^{-1}B^T) \end{bmatrix} \begin{bmatrix} y_{t+1} \\ p_{t+1} \end{bmatrix} = \begin{bmatrix} A - BR^{-1}U^T & 0 \\ -(Q - UR^{-1}U^T & I \end{bmatrix} \begin{bmatrix} y_t \\ p_t \end{bmatrix} \tag{A.10}$$

To complete the solution we require $2n$ boundary conditions for (A.10). Specifying z_0 gives us $n-m$ of these conditions. The remaining condition is the 'transversality condition'

$$\lim_{t\to\infty} \mu_t^T = \lim_{t\to\infty} \lambda^t p_t = 0 \tag{A.11}$$

and the initial condition

$$p_{20} = 0 \tag{A.12}$$

where $p_t^T = \begin{bmatrix} p_{1t}^T & p_{2t}^T \end{bmatrix}$ is partitioned so that p_{1t} is of dimension $(n-m)\times 1$. Equation (A.3), (A.8), (A.10) together with the $2n$ boundary conditions constitute the system under optimal control.

Solving the system under control leads to the following rule

$$w_t = -F \begin{bmatrix} I & 0 \\ -N_{21} & -N_{22} \end{bmatrix} \begin{bmatrix} z_t \\ p_{2t} \end{bmatrix} \tag{A.13}$$

$$\begin{bmatrix} z_{t+1} \\ p_{2t+1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ S_{21} & S_{22} \end{bmatrix} G \begin{bmatrix} I & 0 \\ -N_{21} & -N_{22} \end{bmatrix} \begin{bmatrix} z_t \\ p_{2t} \end{bmatrix} \tag{A.14}$$

$$N = \begin{bmatrix} S_{11} - S_{12}S_{22}^{-1}S_{21} & S_{12}S_{22}^{-1} \\ -S_{22}^{-1}S_{21} & S_{22}^{-1} \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \tag{A.15}$$

$$x_t = -\begin{bmatrix} N_{21} & N_{22} \end{bmatrix} \begin{bmatrix} z_t \\ p_{2t} \end{bmatrix} \tag{A.16}$$

where $F = -(R + B^T SB)^{-1}(B^T SA + U^T), G = A - BF$ and

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{A.17}$$

partitioned so that S_{11} is $(n-m) \times (n-m)$ and S_{22} is $m \times m$ is the solution to the steady-state Ricatti equation

$$S = Q - UF - F^T U^T + F^T RF + \lambda(A - BF)^T S(A - BF) \tag{A.18}$$

The cost-to-go for the optimal policy (OP) at time t is

$$\Omega_t^{OP} = -\frac{1}{2}(\mathrm{tr}(N_{11} Z_t) + \mathrm{tr}(N_{22} p_{2t} p_{2t}^T)) \tag{A.19}$$

where $Z_t = z_t z_t^T$. To achieve optimality the policy-maker sets $p_{20} = 0$ at time $t = 0$. At time $t > 0$ there exists a gain from reneging by resetting $p_{2t} = 0$. It can be shown that $N_{22} < 0$, so the incentive to renege exists at all points along the trajectory of the optimal policy. This is the time-inconsistency problem.

A.2 The Dynamic Programming Discretionary Policy

The evaluate the discretionary (time-consistent) policy we rewrite the cost-to-go Ω_t given by (A.5) as

$$\Omega_t = \frac{1}{2}[y_t^T Q y_t + 2 y_t^T U w_t + w_t^T R w_t + \lambda \Omega_{t+1}] \tag{A.20}$$

The dynamic programming solution then seeks a stationary solution of the form $w_t = -Fz_t$ in which Ω_t is minimized at time t subject to (1) in the knowledge that a similar procedure will be used to minimize Ω_{t+1} at time $t+1$.

Suppose that the policy-maker at time t expects a private-sector response from $t+1$ onwards, determined by subsequent re-optimisation, of the form

$$x_{t+\tau} = -N_{t+1} z_{t+\tau}, \ \tau \geq 1 \tag{A.21}$$

The loss at time t for the *ex ante* optimal policy was from (A.8) found to be a quadratic function of x_t and p_{2t}. We have seen that the inclusion of p_{2t} was the source of the time inconsistency in that case. We therefore seek a lower-order controller $w_t = -Fz_t$ with the cost-to-go quadratic in z_t only. We then write $\Omega_{t+1} = \frac{1}{2} z_{t+1}^T S_{t+1} z_{t+1}$ in (A.20). This leads to the following iterative process for F_t

$$w_t = -F_t z_t \tag{A.22}$$

where

$$\begin{aligned}
F_t &= (\overline{R}_t + \lambda \overline{B}_t^T S_{t+1} \overline{B}_t)^{-1} (\overline{U}_t^T + \lambda \overline{B}_t^T S_{t+1} \overline{A}_t) \\
\overline{R}_t &= R + K_t^T Q_{22} K_t + U^{2T} K_t + K_t^T U^2 \\
K_t &= -(A_{22} + N_{t+1} A_{12})^{-1} (N_{t+1} B^1 + B^2) \\
\overline{B}_t &= B^1 + A_{12} K_t \\
\overline{U}_t &= U^1 + Q_{12} K_t + J_t^T U^2 + J_t^T Q_{22} J_t \\
\overline{J}_t &= -(A_{22} + N_{t+1} A_{12})^{-1} (N_{t+1} A_{11} + A_{12})
\end{aligned}$$

$$\begin{aligned}
\overline{A}_t &= A_{11} + A_{12}J_t \\
S_t &= \overline{Q}_t - \overline{U}_t F_t - F_t^T \overline{U}^T + \overline{F}_t^T \overline{R}_t F_t + \lambda(\overline{A}_t - \overline{B}_t F_t)^T S_{t+1}(\overline{A}_t - \overline{B}_t \overline{F}_t) \\
\overline{Q}_t &= Q_{11} + J_t^T Q_{21} + Q_{12}J_t + J_t^T Q_{22} J_t \\
N_t &= -J_t + K_t F_t
\end{aligned}$$

where $B = \begin{bmatrix} B^1 \\ B^2 \end{bmatrix}$, $U = \begin{bmatrix} U^1 \\ U^2 \end{bmatrix}$, $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, and Q similarly are partitioned conformably with the predetermined and non-predetermined components of the state vector.

The sequence above describes an iterative process for F_t, N_t, and S_t starting with some initial values for N_t and S_t. If the process converges to stationary values, F, N and S say, then the time-consistent feedback rule is $w_t = -Fz_t$ with loss at time t given by

$$\Omega_t^{TC} = \frac{1}{2} z_t^T S z_t = \frac{1}{2} \mathrm{tr}(S Z_t) \tag{A.23}$$

A.3 Optimized Simple Rules

We now consider simple sub-optimal rules of the form

$$w_t = D y_t = D \begin{bmatrix} z_t \\ x_t \end{bmatrix} \tag{A.24}$$

where D is constrained to be sparse in some specified way. Rule can be quite general. By augmenting the state vector in an appropriate way it can represent a PID (proportional-integral-derivative)controller (though the paper is restricted to a simple proportional controller only).

Substituting (A.3) into (A.5) gives

$$\Omega_t = \frac{1}{2} \sum_{i=0}^{\infty} \lambda_t y_{t+i}^T P_{t+i} y_{t+i} \tag{A.25}$$

where $P = Q + UD + D^T U^T + D^T RD$. The system under control (A.1), with w_t given by (A.3), has a rational expectations solution with $x_t = -N z_t$ where $N = N(D)$. Hence

$$y_t^T P y_t = z_t^T T z_t \tag{A.26}$$

where $T = P_{11} - N^T P_{21} - P_{12} N + N^T P_{22} N$, P is partitioned as for S in (A.17) onwards and

$$z_{t+1} = (G_{11} - G_{12} N) z_t \tag{A.27}$$

where $G = A + BD$ is partitioned as for P. Solving (A.27) we have

$$z_t = (G_{11} - G_{12} N)^t z_0 \tag{A.28}$$

Hence from (A.29), (A.26) and (A.28) we may write at time t

$$\Omega_t^{SIM} = \frac{1}{2} z_t^T V z_t = \frac{1}{2}\text{tr}(V Z_t) \tag{A.29}$$

where $Z_t = z_t z_t^T$ and V satisfies the *Lyapunov* equation

$$V = T + H^T V H \tag{A.30}$$

where $H = G_{11} - G_{12}N$. At time $t = 0$ the optimized simple rule is then found by minimizing Ω_0 given by (A.29) with respect to the non-zero elements of D given z_0 using a standard numerical technique. An important feature of the result is that unlike the previous solution the optimal value of D, D^* say, is not independent of z_0. That is to say

$$D^* = D^*(z_0)$$

A.4 The Stochastic Case

Consider the stochastic generalization of (A.1)

$$\begin{bmatrix} z_{t+1} \\ x^e_{t+1,t} \end{bmatrix} = A \begin{bmatrix} z_t \\ x_t \end{bmatrix} + Bw_t + \begin{bmatrix} u_t \\ 0 \end{bmatrix} \tag{A.31}$$

where u_t is an $n \times 1$ vector of white noise disturbances independently distributed with $\text{cov}(u_t) = \Sigma$. Then, it can be shown that certainty equivalence applies to all the policy rules apart from the simple rules (see Currie and Levine (1993)). The expected loss at time t is as before with quadratic terms of the form $z_t^T X z_t = \text{tr}(X z_t, Z_t^T)$ replaced with

$$\mathcal{E}_t\left(\text{tr}\left[X\left(z_t z_t^T + \sum_{i=1}^{\infty} \lambda^t u_{t+i} u_{t+i}^T\right)\right]\right) = \text{tr}\left[X\left(z_t^T z_t + \frac{\lambda}{1-\lambda}\Sigma\right)\right] \tag{A.32}$$

where $\mathcal{E}_t$ is the expectations operator with expectations formed at time t.

Thus for the optimal policy with commitment (A.19) becomes in the stochastic case

$$\Omega_t^{OP} = -\frac{1}{2}\text{tr}\left(N_{11}\left(Z_t + \frac{\lambda}{1-\lambda}\Sigma\right) + N_{22} p_{2t} p_{2t}^T\right) \tag{A.33}$$

For the time-consistent policy (A.23) becomes

$$\Omega_t^{TC} = -\frac{1}{2}\text{tr}\left(S\left(Z_t + \frac{\lambda}{1-\lambda}\Sigma\right)\right) \tag{A.34}$$

and for the simple rule, generalizing (A.29)

$$\Omega_t^{SIM} = -\frac{1}{2}\text{tr}\left(V\left(Z_t + \frac{\lambda}{1-\lambda}\Sigma\right)\right) \tag{A.35}$$

The optimized simple rule is found at time $t = 0$ by minimizing Ω_0^{SIM} given by (A.35). Now we find that

$$D^* = D^*\left(z_0 + \frac{\lambda}{1-\lambda}\Sigma\right) \tag{A.36}$$

or, in other words, the optimized rule depends both on the initial displacement z_0 and on the covariance matrix of disturbances Σ.

B Estimation Results

Table1 : Priors for Baseline model

	Distribution	Mean	Standard Deviation	Percentiles 1%	99%
ρ_i	B	0.75	0.15	0.538	0.981
θ	G	1.7	0.5	1.099	3.074
χ	G	0.15	0.1	0.044	0.473
φ	G	1.75	0.5	1.148	3.118
σ	G	1.5	0.8	0.609	3.952
γ	B	0.7	0.1	0.566	0.897
h	B	0.7	0.1	0.566	0.897
ρ_a	B	0.7	0.15	0.492	0.959
ρ_g	B	0.7	0.15	0.492	0.959
π^*	G	4	2	1.745	10.045
r*	G	2	1	0.872	5.023
sd_g	IG1	1.7	inf	0.635	9.260
sd_e	IG1	1	inf	0.372	5.699
sd_a	IG1	1.7	inf	0.635	9.260

For all models, g is calibrated to 0.22. Distributions: G (Gamma), B (Beta), and), IG1 (Inverse Gamma-1). ρ corresponds to the autoregressive coefficient of an AR(1) process. sd stands for the standard deviation of the shocks. .Last two columns, report the inverse cumulative distribution function of each prior ordinate for thepercentiles 0.01 and 0.99.

Table 2: Parameter estimates for j=1

Model	**Model G** ($\gamma = 0$)			**Model H** (h = 0)			**Model GH** (γ = h= 0)			**Model Z**		
	Posterior Distribution			Posterior Distribution			Posterior Distribution			Posterior Distribution		
Coefficient	Median	10%	90%	Median	10%	90%	Median	10%	90%	Median	10%	90%
ρ_i	0.80	[0.73	, 0.85]	0.67	[0.55	, 0.75]	0.72	[0.64	, 0.79]	0.77	[0.71	, 0.83]
θ	2.55	[2.15	, 2.99]	2.68	[2.18	, 3.29]	2.64	[2.14	, 3.20]	2.25	[1.86	, 2.58]
χ	0.16	[0.08	, 0.30]	0.71	[0.49	, 0.98]	0.27	[0.14	, 0.44]	0.47	[0.25	, 0.74]
φ	1.46	[1.06	, 1.94]	2.40	[1.70	, 3.15]	2.16	[1.47	, 2.77]	1.32	[0.99	, 1.81]
σ	3.29	[2.28	, 4.98]	2.50	[1.47	, 4.29]	3.91	[2.86	, 5.76]	3.23	[2.55	, 4.21]
γ				0.59	[0.45	, 0.72]				0.54	[0.40	, 0.68]
h	0.85	0.74	, 0.91							0.85	[0.73	, 0.92]
ρ_a	0.91	[0.85	, 0.94]	0.91	[0.87	, 0.94]	0.90	[0.85	, 0.93]	0.94	[0.89	, 0.96]
ρ_g	0.92	[0.88	, 0.95]	0.91	[0.88	, 0.93]	0.90	[0.87	, 0.93]	0.93	[0.89	, 0.96]
π^*	3.00	[2.43	, 3.60]	2.72	[2.14	, 3.32]	2.96	[2.40	, 3.51]	2.88	[1.75	, 3.55]
r*	1.86	[1.38	, 2.49]	1.82	[1.26	, 2.48]	1.90	[1.33	, 2.46]	1.80	[1.25	, 2.38]
sd_g	2.19	[1.99	, 2.39]	3.23	[2.53	, 4.17]	2.75	[2.35	, 3.11]	2.23	[2.05	, 2.47]
sd_e	0.17	[0.15	, 0.19]	0.20	[0.18	, 0.23]	0.17	[0.16	, 0.19]	0.19	[0.17	, 0.21]
sd_a	0.78	[0.61	, 1.01]	0.51	[0.42	, 0.64]	0.59	[0.48	, 0.74]	0.72	[0.57	, 0.93]

Median and posterior deciles of the draws generated with a Random Walk Metropolis algorithm. Discarded the first 30,000 draws, retained the remaining 100,000 values

Table 3: Model Comparisons

Reversible Jump MCMC

Model	*Posterior Odds, P(m | data)*
G (γ = 0)	0.56
H (h= 0)	0.09
GH (γ =h= 0)	0.03
Z	0.32

Reversible MCMC of Dellaportas et al. (2002). 100,000 draws to obtain the proposal densities. For the Metropolis step, Discarded the first 20,000 values and retained the remaining 180,000. Posterior odds P(m|data) based on assigning each model equal prior probability. Model proposal density assigns equal probability to the jump to any of four possible models, regardless of the current model in the chain.